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Repr. of finite groups

Talk 1, Day 1

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G finite group, k field

1. The group algebra

$kG := \{ \sum_{g\in G} \alpha_g g \mid \alpha_g \in k \}$ as k-VS, kG algebra with mult. the gp actn

kG-mod := cat. of fg. left modules over kG

Ex: $k \in kG$-mod, $g\cdot\alpha = \alpha$ $(\alpha \in k)$.

Thm (Krull-Schmidt)

$\forall M \in kG\text{-mod} \Rightarrow M = M_1 \oplus \cdots \oplus M_n$, M_i indec kG-modules

unique up to isos & reordering!

Thm (Maschke)

kG is semi-simple iff $\operatorname{char} k \nmid |G|$ = order of G.

Idea: If $\operatorname{char} k \nmid |G|$, $M \in kG$-mod, $N \subseteq M$

claim: N is a direct summand: $\pi' \in \mathrm{End}(M)$ s.t. $\pi(M) = N$

$$\rightsquigarrow \pi := \frac{1}{|G|} \sum_{g\in G} g\pi' g^{-1} \in \mathrm{End}_{kG}(M) \rightsquigarrow M = N \oplus (id - \pi)\cdot M. \quad \square$$

2. Monoidal structure

Tensor product: $kG\text{-mod} \times kG\text{-mod} \longrightarrow kG\text{-mod}$

$M, N \longmapsto M \otimes N := M \otimes_k N$ as VS/k

$g(m\otimes n) := gm \otimes gn$

(Comes from Hopf alg. str. on kG, with comult. $\Delta(g) = g \otimes g$.)

Example: $k \otimes M \cong M \cong M \otimes k$ $\rightsquigarrow$ k is "tensor unit".

Internal hom: $(kG\text{-mod})^{op} \times kG\text{-mod} \longrightarrow kG\text{-mod}$

$(M, N) \longmapsto \mathrm{Hom}(M,N) := \mathrm{Hom}_k(M,N)$ as VS/k

$(g\cdot f)(m) \mapsto g\cdot f(g^{-1}m)$

special case,

Dual: $(kG\text{-mod})^{op} \longrightarrow kG\text{-mod}$

$M \longmapsto M^* = \mathrm{Hom}(M, k)$, with $(g\cdot f)(m) = f(g^{-1}m)$.

⇝ $(kG, \otimes, k)$ is a closed monoidal category, i.e

tensoring is left adj. to internal Hom:

$$(-\otimes M) \dashv \mathrm{Hom}(M,-) \;:\Longleftrightarrow\; \mathrm{Hom}_{kG}(L\otimes M, N) \cong \mathrm{Hom}_{kG}(L, \mathrm{Hom}(M,N))$$

<u>Theorem</u>: $kG \cong kG^*$, kG is Frobenius: ← cuz. symmetric algebra

$\exists\; kG\times kG \to k$ non-deg bilin, s.t. $(\ell\cdot m, n) = (\ell, m\cdot n)$

In fact, already def. on G:

$$\sigma : kG\times kG \to k$$

$$(g,h) \mapsto \delta_{gh^{-1}} \quad ⇝ \quad kG \xrightarrow{\cong} kG^* = \mathrm{Hom}(kG,k)$$

$$m \mapsto \sigma(m,-)$$

Rmk:
- kG is self-injective, i.e injective as kG-module.
- every inj/proj. kG-module is proj./inj.
- P proj. indec. kG-module $\Rightarrow$ $P/\mathrm{rad}\,P \cong \mathrm{soc}(P)$ (because kG symm. alg.)

3. Restriction & induction

$H \subset G$ a subgroup, $M \in kG\text{-mod}$ ⇝ $\overset{M_H}{\overset{\shortparallel}{M}} \in kH\text{-mod}$ just restrict the action!

$L \in kH\text{-mod}$ ⇝ $L^G := kG \otimes_H L \in kG\text{-mod}$, action by left mult on kG

Get:

$$\left.\begin{array}{l} \mathrm{Res}^G_H : kG\text{-mod} \to kH\text{-mod} \\ \qquad\qquad M \longmapsto M_H \\ \\ \mathrm{Ind}^G_H : kH\text{-mod} \to kG\text{-mod} \\ \qquad\qquad L \longmapsto L^G \end{array}\right\}\ \text{exact functors}$$

Thm. (Frobenius reciprocity) Ind ⊣ Res (in general), and Res ⊣ Ind (because finite gp. case).

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4. Resolutions of modules

$M \in kG\text{-mod}$ $\quad P_\bullet = \cdots P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} P_{n-2} \to \cdots \to P_0 \xrightarrow{\varepsilon} M \to 0$

Ex: M, $P_0 := P_M$ projective cover of M

$P_1 := P_{\ker(\varepsilon)}$, ..., $P_i = P_{\ker(\partial_{i-1})}$, proj. covers.

$$\begin{tikzcd}
\cdots & P_{\ker\partial} \arrow[rr, dashed] \arrow[dr] & & P_1 \arrow[rr, dashed, "\partial_1"] \arrow[dr, two heads] & & P_M \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
& & \ker\partial_1 \arrow[ur] & & \ker\varepsilon \arrow[ur] & & &
\end{tikzcd}$$

Dual: $I^\bullet = \quad 0 \to M \xrightarrow{\eta} I^1 \to I^2 \to I^3 \to \cdots$

Ex: M, $I^1 = E_M$ = inj. hull of M, etc.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\eta"] & E_M \arrow[rr, dashed, "\partial_1"] \arrow[dr] & & E \arrow[rr, dashed] \arrow[dr] & & \cdots \\
& & & \operatorname{coker}\eta \arrow[ur] & & \operatorname{coker}\partial^1 \arrow[ur] &
\end{tikzcd}$$

Can splice together the two:

$$\begin{tikzcd}
& & & & & & & I^1 \arrow[d, equal] & & I^2 \arrow[d, equal] & & \\
\cdots \arrow[r] & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[rr, dashed, "\partial_0 :="] \arrow[dr, "\varepsilon"'] & & P_{-1} \arrow[rr, "\partial_{-1}"] \arrow[dr] & & P_{-2} \arrow[rr, "\partial_{-2}"] \arrow[dr] & & \cdots \\
& & \ker\partial_1 \arrow[ur] \arrow[dd, equal] & & \ker\partial_0 \arrow[ur] \arrow[dd, equal] & & M \arrow[ur, "\eta"'] \arrow[dr] & & \operatorname{coker}\eta \arrow[ur] \arrow[dd, equal] & & \operatorname{Coker}\partial_{-1} \arrow[dd, equal] & \\
& & & & & 0 \arrow[ur] & & 0 & & & & \\
\text{Syzygies :} & & \Omega^2(M) & & \Omega^1(M) & & & & \Omega^{-1}(M) & & \Omega^{-2}(M) & \cdots
\end{tikzcd}$$

5. Group cohomology

$M \in kG\text{-mod}$, $P_\bullet$ a proj. resolution : $\cdots \to P_2 \to P_1 \to P_0 \to 0$

$N \in kG\text{-mod}$, apply $\operatorname{Hom}_{kG}(-,N)$ to it:

$$0 \to \operatorname{Hom}_{kG}(P_0,N) \to \operatorname{Hom}_{kG}(P_1,N) \to \cdots \xrightarrow{\delta_{n-1}} \operatorname{Hom}_{kG}(P_n,N) \xrightarrow{\delta_n}$$

not exact anymore:

$$\operatorname{Ext}^n_{kG}(M,N) := H^n(\operatorname{Hom}_{kG}(P_\bullet,N)) := \ker\delta_n / \operatorname{Im}\delta_{n-1}$$

Rmk: $\uparrow$ does not depend on the resolution

For $M = k$: $H^n(G,N) := \mathrm{Ext}^n_{kG}(k,N)$: cohomology of G with coeff. in N.

Examples : • $H^0(G,N) = \mathrm{Ext}^0_{kG}(k,N) = \mathrm{Hom}_{kG}(k,N) = N^G$

• $N = k \rightsquigarrow H^n(G,k) = \mathrm{Ext}^n_{kG}$ ↗ invariants of G-action on N.

the cohomology of G."

• Ring structure on $H^n(G,k) := \mathrm{Ext}^n_{kG}(k,k)$:

Consider $[0 \to k \to M_1 \to \cdots \to M_n \to k \to 0] \in \mathrm{Ext}^n_{kG}(k,k)$

$[0 \to k \to N_1 \to \cdots \to N_m \to k \to 0] \in \mathrm{Ext}^m_{kG}(k,k)$

splice them together:

$$\begin{tikzcd}
{[0} \arrow[r] & k \arrow[r] & M_1 \arrow[r] & \cdots \arrow[r] & M_n \arrow[rr, dashed] \arrow[dr] & & N_1 \arrow[r] & \cdots \arrow[r] & N_m \arrow[r] & k \arrow[r] & {0]} \in \mathrm{Ext}^{n+m}_{kG}(k,k) \\
& & & & & k \arrow[ur] \arrow[dr] & & & & & \\
& & & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

$\wr$

get a graded ring structure on $H^*(G,k)$.

• In the same way, by splicing get a (right) graded module structure of $H^*(G,k)$ on $H^*(G,M)$:

$$0 \to M \to M_1 \to \cdots \to M_n \to \underbrace{k \to 0 \quad 0 \to k}_{} \to N_1 \to \cdots N_m \to k \to 0$$

$$H^{n+m}(G,M) \ni \varepsilon$$

<u>Ex</u>: char $k = p > 0$, C_p cyclic group of order p

$\langle g \rangle = C_p$, g a generator

$$\rightsquigarrow \quad kC_p = \left\{ \sum_{i=1}^{p} \alpha_i g^i \;\middle|\; \alpha_i \in k \right\} \cong k[x]/\underbrace{(x^p - 1)}_{= (x-1)^p} \overset{\overset{y := x-1}{\downarrow}}{\cong} k[y]/y^2$$

Take $p=2$: $kG \overset{(\sim)}{=} k[y]/y^2$. Resolution of k:

$$\begin{tikzcd} \cdots \arrow[r, "\cdot y"] & \frac{k[y]}{y^2} \arrow[r, "y\cdot"] & \frac{k[y]}{y^2} \arrow[r, "\cdot y"] & k \arrow[r] & 0 \end{tikzcd}$$

$\rightsquigarrow \mathrm{Hom}_{kG}(-,k)$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{kG}(kG,k) \arrow[r, "0"] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Hom}_{kG}(kG,k) \arrow[r, "0"] \arrow[d, phantom, sloped, "\cong"] & {} \\ 0 \arrow[r] & k \arrow[r, "0"] & k \arrow[r, "0"] & \cdots \end{tikzcd}$$

$\Rightarrow \quad H^n(G,k) \cong k \quad \forall n \geq 0$.

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